Phys 410
Fall 2015
Lecture #11 Summary
6 October, 2015

We continued to consider the motion of the Foucault pendulum. The z-motion of the bob is fairly simple, essentially reducing to the statement that $\mathrm{T} \cong \mathrm{mg}$. The tension in the horizontal xy-plane is $\mathrm{T}_x = -mgx/L$, and $\mathrm{T}_y = -mgy/L$. The Coriolis force is found from the cross product $2m\dot{\vec{r}} \times \vec{\Omega}$. We write $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$ and $\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta)$. After carrying out the cross product and putting the results into the equation of motion, broken down into components, we get: $m\ddot{x} = -\frac{mgx}{L} + 0 + 2m(\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta)$, and $m\ddot{y} = -\frac{mgy}{L} + 0 - 2m\dot{x}\Omega\cos\theta$. We shall drop the $\dot{z}$ term in the x-equation because it is the product of two small velocities, define the constants $\omega_0^2 \equiv g/L$, and $\Omega_z \equiv \Omega\cos\theta$, to get two coupled equations of motion:

$$\ddot{x} - 2\dot{y}\Omega_z + \omega_0^2 x = 0$$

$$\ddot{y} + 2\dot{x}\Omega_z + \omega_0^2 y = 0$$

The first and third terms alone would give un-coupled simple harmonic motion in the xy-plane. The coupling terms look like a form of dissipation (of the form $F_{dis} = -bv$) but in fact they represent a coupling of energy from one direction of motion to the other. The energy in the oscillations sloshes back and forth between x and y.

These equations can be combined in a manner similar to the equations for motion of a charged particle in a magnetic field. Take the first equation plus "i" times the second equation, and define the new dependent complex variable $\eta(t) \equiv x(t) + iy(t)$ to get a single equation: $\ddot{\eta} + i2\dot{\eta}\Omega_z + \omega_0^2\eta = 0$. Trying a solution of the form $\eta(t) = e^{-i\alpha t}$, we get an auxiliary equation with solutions $\alpha = \Omega_z \pm \sqrt{\omega_0^2 + \omega\Omega_z^2}$. Using the fact that the pendulum oscillates many times compared to the rotation period of the Earth (i.e. $\omega_0 \gg \Omega_z$) we come to the solution $\eta(t) = e^{-i\Omega_z t}\left(C_1 e^{-i\omega_0 t} + C_2 e^{+i\omega_0 t}\right)$. To supply initial conditions, consider pulling the pendulum bob to a displacement A in the east (x) direction ($y = 0$) and release it from rest. In this case one finds $C_1 = C_2 = A/2$, and the solution is $\eta(t) = Ae^{-i\Omega_z t}\cos(\omega_0 t)$. Taking the real and imaginary parts to get the actual equations of motion in real space gives $x(t) = A\cos(\Omega_z t)\cos(\omega_0 t)$ and $y(t) = -A\sin(\Omega_z t)\cos(\omega_0 t)$. The pendulum swings back and forth on a short time scale, described by the factor of $\cos(\omega_0 t)$. On longer time scales, the plane of oscillation rotates, as described by the factors of $\cos(\Omega_z t)$ and $-\sin(\Omega_z t)$, with $\omega_0 \gg \Omega_z$. This slow rotation of the plane of oscillation occurs at a frequency that depends on your (co-)latitude on the Earth $\Omega_z \equiv \Omega\cos\theta$, where the rotation frequency of the Earth is $\Omega = 7 \times 10^{-5}$ Rads/s.

Lagrangian mechanics is a different way of looking at things. We imagine that the starting and ending conditions of the particle or system are perfectly specified. We consider every possible evolution of the system that starts at the initial location at the initial time and ends up at the final location at the final time. We define a "physics-based" cost function that can be evaluated for every possible evolution of the particle. We then identify the trajectory (or time evolution of the system) that minimizes this cost function, and that is the evolution that nature chooses. Later we will present Hamilton's principle which states that the cost function is the difference between the time-averaged kinetic energy and time-averaged potential energy. Finding the minimum of this cost function is a problem in the calculus of variations.

The calculus of variations is used to find extremum values of integral functionals (a function of a function). An example is a calculation of the shortest distance between two points in a plane. One can write the distance in terms of an integral over the path from the designated starting point (x_1, y_1) to the designated end point (x_2, y_2) as $L = \int_1^2 ds = \int_1^2 \sqrt{dx^2 + dy^2}$. If we (arbitrarily) treat the x coordinate as the independent variable we can write the integral as $L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$, where we have written $(dy/dx)^2$ as $(y')^2$. Our objective is to find the path $y(x)$ that minimizes this integral. This is a problem in the calculus of variations.

A second example is Fermat's principle. This is the problem of how light rays propagate from point 1 to point 2 through a variable dielectric medium characterized by an index of refraction that varies with position in a plane as $n(x, y)$. The light moves with variable speed $v = c/n(x, y)$. Fermat's principle says that light will take the path that minimizes the time to travel between the two points: $time(1 \to 2) = \frac{1}{c}\int_{x_1}^{x_2} n(x, y)\sqrt{1 + (y')^2}\, dx$. Again we need to find the path $y(x)$ that minimizes this integral. This is another problem in the calculus of variations.

The Euler-Lagrange equation is derived by assuming that there is an infinite family of "wrong" trajectories between points 1 and 2 parameterized by the arbitrary error function $\eta(x)$ and the constant α as $Y(x) = y(x) + \alpha\eta(x)$, where $y(x)$ is the "true" trajectory that we wish to find. The objective is to minimize (or more generally, to make stationary) the integral $S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\, dx$, and this will be accomplished by taking $dS/d\alpha$ and setting it equal to zero. The result, after integrating by parts, is that the following expression must be satisfied for all points $x_1 \leq x \leq x_2$: $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$, called the Euler-Lagrange equation.